

The Complex Common Product and Fractals

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Abstract

Fractals are abstract self-affine structures with several correlates in the physical world including ferns, trees, corals, etc. Fractal sets such as Mandelbrot and Julia rely on complex arithmetic, including respective products and magnitude. In this work, we suggest that the common product can also be used in complex arithmetics, and illustrate this possibility regarding its effect on obtaining variations of the Mandelbrot set.

‘Starshine, could it also be a fractal?’

LdaFC

with:

$$s_x = \text{sign}(x) \quad (2)$$

$$s_y = \text{sign}(y) \quad (3)$$

$$s_{xy} = s_x s_y \quad (4)$$

1 Introduction

Complex arithmetic and algebra (e.g. [1, 2]), developed mainly by the Italian mathematician Rafael Bombelli (1526–1572), involves the imaginary number $i = \sqrt{-1}$, therefore allowing a respective space that has more structure than the real counterpart.

The *common product* has been introduced as a manner to extend the Jaccard and other similarity indices to real, including possibly negative, values [3, 4, 5]. Several interesting related results have been obtained, including the real-valued Jaccard and coincidence indices, which have been shown to provide enhanced properties in pattern recognition [6], complex networks [7], and hierarchical clustering [8].

In this work, we propose a complex correlate of the common product and illustrates its effect on the calculation of a respective variation of the Mandelbrot set (e.g. [9, 10]).

2 The Real and Complex Common Product

Given any two real values x and y , their product can be expressed as xy , while their respective *common product* is given as:

$$x \circ y = s_{xy} \min \{s_x x, s_y y\} \quad (1)$$

In the case of two complex numbers $z = x + iy$ and $w = r + is$, their traditional complex product corresponds to:

$$zw = xr - ys + i[xs + ry] \quad (5)$$

The *complex common product* can be defined as:

$$\begin{aligned} z \circ w = & s_{xr} \min(s_x x, s_r r) - s_{ys} \min(s_y y, s_s s) \\ & + i[s_{xs} \min(s_x x, s_s s) + s_{ry} \min(s_r r, s_y y)] \end{aligned} \quad (6)$$

The traditional magnitude (or modulus) of a complex number z is as follows:

$$|z| = \sqrt{xx^*} \quad (7)$$

where z^* is the complex conjugate of z .

The common product-based magnitude of a complex number z can be expressed as:

$$|z|_c = \sqrt{x \circ x^*} \quad (8)$$

3 Common Product-Based Mandelbrot Set

The classic Mandelbrot set [9] generation involves an iterated product, with the modulus convergence defining the respective intensity or color of the respectively obtained image. The common product correlate of the Mandelbrot

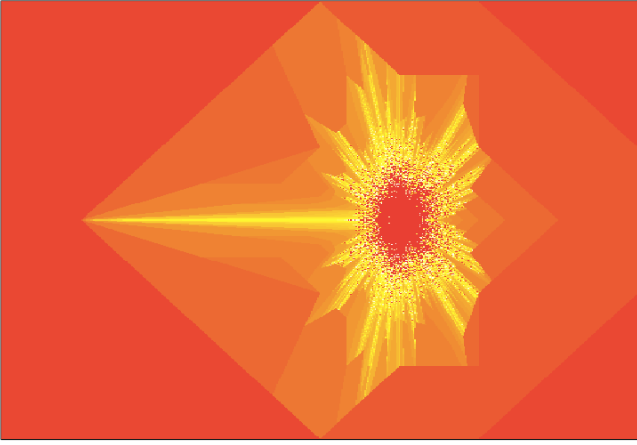


Figure 1: The Mandelbrot set correlate obtained by using the complex common product.

set, shown in Figure 1 was obtained by employing the complex common product and respective absolute value instead of the traditional complex product and modulus.

The classic Mandelbrot set is shown in Figure 2 for comparison purposes.

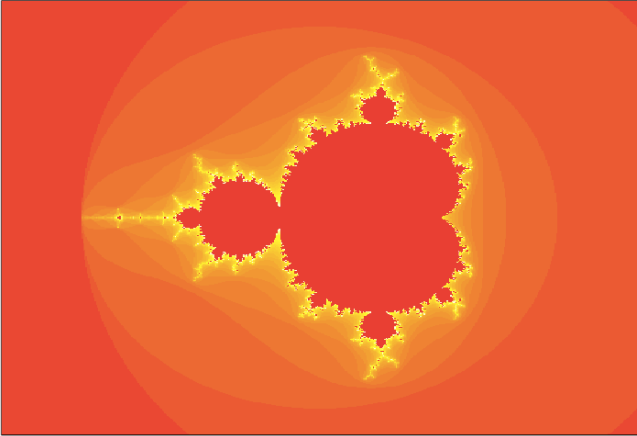


Figure 2: The Mandelbrot set correlate obtained by using the complex common product.

While the overall bilateral symmetry is maintained with respect to the classic Mandelbrot set, the obtained variant presents much sharper detailed structure that resembles a starshine or comet profile. These strong structural modifications are a consequence of the common product being more strict regarding the quantification of similarity between two numbers [5].

4 Concluding Remarks

In this brief work, we described the complex correlate of the common product, and then applied it to obtain a vari-

ation of the classic Mandelbrot set. Though maintaining the original bilateral symmetry, the new structure is characterized by much sharper rays projecting from the main body.

The concepts and results presented in this work paves the way to several related further developments.

As far as the proposal of the complex correlate of the common product is concerned, it would be interesting to re-assess several concepts, methods, and applications that employ the classic complex product in the light of this alternative possibility. One particularly interesting prospect is to re-evaluate the Riemann Zeta function (e.g. [11]) from the perspective of the complex common product.

From the perspective of fractals, further works include a more systematic study of the fractal properties of the modified Mandelbrot set, application to other fractal structures, and characterization of the obtained structure by using, for instance, the fractal dimension.

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